63-3-2

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Contract No.: DA-11-022-ORD-2059

ON THE STABILITY OF A MACLAURIN
SPHEROID OF SMALL VISCOSITY

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ABSTRACT

The stability of a viscous Maclaurin spheroid is solved asymptotically for small kinematic viscosity, ν . It is shown that, in this limit, the frequency of oscillation, n, with respect to the mode which becomes neutrally stable in the absence of viscosity at the point of bifurcation (where the eccentricity, e, of the meridional section is approximately 0.8127), is

$$n = n_0 + i \frac{5\nu n_0^2}{\varphi(e)} + 0(\nu) \tag{1}$$

In the foregoing formula n_0 denotes the frequency in the absence of viscosity, a the radius of the equational section and $\varphi(e)$ a certain function of e which changes sign at e = 0.8127 and is positive for smaller values of e . From equation (1) it follows that the Maclaurin spheroid is indeed unstable beyond the point of bifurcation when viscosity is present.

ON THE STABILITY OF A MACLAURIN SPHEROID OF SMALL VISCOSITY

P. H. Roberts*

and

K. Stewartson**

1. INTRODUCTION.

It is well known that, for an incompressible fluid mass in a state of

uniform rotation and in equilibrium under its own gravitation, a spheroid of

revolution is an exact figure of equilibrium. This is the spheroid of Maclaurin.

The stability of these spheroids with respect to infinitesimal perturbations has

been investigated in a number of classical papers by Riemann (1860), Bryan

(1888) and Cartan (1922) (see also Lebovitz, [1961]). From these investigations

it is known that, in the absence of viscosity, the Maclaurin spheroids are

stable provided that the eccentricity, e, of the meridional section is less than

0.9529; and that, when the eccentricity exceeds this limit, the Maclaurin

spheroids are unstable by oscillations of increasing amplitude (the period of

oscillation being, in fact, equal to the angular velocity Ω). However, it

was known, even before the upper limit e = 0.9529 was established by Bryan,

that the Maclaurin spheroid allows a neutral mode of oscillation at e = 0.8127,

where the ellipsoids of Jacobi branch off.

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The occurrence of a neutral mode for the Maclaurin spheroids at the point of bifurcation is suggestive; indeed, it was categorically stated by Thomson and Tait (1883, p. 333) that "the equilibrium in the revolutional figure is stable or unstable according as e is less than or greater than 0.81266." In definitely associating the occurrence of the neutral mode (for infinitesimal oscillations) with the onset of instability, Thomson and Tait clearly had the role of viscosity in mind. They state, for example, "if there be any viscosity, however slight, in the liquid or if there be any imperfectly elastic solid, however small, floating on it or sunk within it, the equilibrium in any case of energy either minimax or a maximum cannot be secularly stable: the only secularly stable configurations are those in which the energy is the minimum with given momentum." All these considerations became, of course, clarified by the investigations of Poincare which showed that, at the point of bifurcation of the Maclaurin sequence, it is possible to deform the Maclaurin spheroid into a body of the same angular momentum but of lower total energy. From this last fact one generally concludes that, should any dissipative mechanism be operative, then it would indeed operate and induce the transition to the state of lower energy, i.e. instability will arise. In other words, the belief and the conjecture is that, were one to investigate the stability of a _viscous_ Maclaurin spheroid (again for infinitesimal perturbations) then one would find that it becomes unstable (with an initial infinitisemal amplitude increasing exponentially with time) when one surpasses the point of bifurcation. This last argument is usually illustrated by simple mechanical examples (cf. Jeans, [1929], p. 201). While

the suggestion that the objects should be unstable under the circumstances envisaged is very reasonable, it is not a forgone conclusion that motions consistent with the equations of hydrodynamics do exist which will allow instability: examples are known when a normal mode analysis proves stability in circumstances when an energy argument would suggest instability (cf., Chandrasekhar, 1961, S103). And finally, the energy argument provides no basis for estimating the growth rates of the unstable mode when viscosity is operative. For these reasons, we shall consider in this paper the stability of the Maclaurin spheroid allowing for viscosity in the limit when it is small; the assumption of small viscosity is made, for the problem is then tractable by a boundary layer argument.

II.   THE EQUILIBRIUM STATE

Let (x, y, z) be cartesian co-ordinates in which the z-axis is parallel to $\underset{\sim}{\Omega}$, the angular velocity of the spheroid, and in which the origin is at the center of that body. The equation of the surface, S_e, of the spheroid is then of the form

$$\frac{\widetilde{\omega}^2}{a^2} + \frac{z^2}{c^2} = 1 \; , \tag{2.1}$$

where $\widetilde{\omega} = (x^2 + y^2)^{\frac{1}{2}}$. It is convenient, for the later discussion, to introduce oblate spheroidal co-ordinates here. For this purpose, define k and E by

$$k = a^2 - c^2, \quad E = c/k \; . \tag{2.2}$$

The following relations then exist between a, c, k, E and the eccentricity of the spheroid, e :

$$a = k(1 + E^2)^{\frac{1}{2}}, \quad c = kE = a(1 - e^2)^{\frac{1}{2}} \; ,$$

$$k = ae, \quad\quad\quad \sin^{-1} e = \cot^{-1} E \quad . \tag{2.3}$$

Oblate spheroidal co-ordinates (ζ, θ, φ) are defined by

$$x = k(1 + \zeta^2)^{\frac{1}{2}} \sin \theta \cos \varphi \; ,$$

$$y = k(1 + \zeta^2)^{\frac{1}{2}} \sin \theta \sin \varphi \; , \tag{2.4}$$

$$z = k \zeta \cos \theta \; ,$$

$(0 < \ < \infty, \ 0 < \theta < \pi, \ 0 < \varphi < 2\pi)$. They form an orthogonal curvilinear system,

right-handed in the order (ζ, θ, φ), and having scale factors h_ζ h_θ and h_φ

given by

$$h_\zeta = k\left(\frac{\zeta^2 + \mu^2}{1 + \zeta^2}\right)^{\frac{1}{2}} , \quad h_\theta = k(\zeta^2 + \mu^2)^{\frac{1}{2}} , \quad h_\varphi = k(1 + \zeta^2)^{\frac{1}{2}}(1 - \mu^2)^{\frac{1}{2}} , \quad (2.5)$$

where $\mu = \cos\theta$. The surfaces of equal ζ are a confocal system of spheroids

and, in virtue of the definitions (2.2), S_e is the particular member

$$\zeta = E \qquad\qquad\qquad\qquad (2.6)$$

of this system.

Since the material of which the spheroid is composed is supposed to be

incompressible and of uniform density ρ, the equation of hydrostatic

equilibrium is simply

$$\frac{p}{\rho} - V - \frac{1}{2}\Omega^2 \widetilde{\omega}^2 = \text{constant} , \qquad\qquad (2.7)$$

where p is the pressure, and V is the gravitational potential. Now the

potential within the oblate spheroid (2.1) is given by (see, for example, Routh,

1922, Art. 211)

$$\frac{V}{\pi G\rho a^2 c} = I - A_1 \widetilde{\omega}^2 - A_3 z^2 , \qquad\qquad (2.8)$$

where I, A_1 and A_3 are constants given by

$$I = \int_0^\infty \frac{du}{(a^2+u)(c^2+u)^{\frac{1}{2}}} = \frac{2}{ae} \sin^{-1} e \ ,$$

$$A_1 = \int_0^\infty \frac{du}{(a^2+u)^2 (c^2+u)^{\frac{1}{2}}} = \frac{1}{(ae)^3}[\sin^{-1} e - e(1-e^2)^{\frac{1}{2}}] \ , \qquad (2.9)$$

$$A_3 = \int_0^\infty \frac{du}{(a^2+u)(c^2+u)^{\frac{3}{2}}} = \frac{2}{(ae)^3}[\frac{e}{(1-e^2)^{\frac{1}{2}}} - \sin^{-1} e] \ .$$

Thus, equation (2.7) may be rewritten in the form

$$\frac{p}{\rho} = \text{constant} - \pi G\rho a^2 c[(A_1 - \frac{\Omega^2}{2\pi G\rho a^2 c})\widetilde{\omega}^2 + A_3 z^2] \ . \qquad (2.10)$$

However, since p must vanish everywhere on the equilibrium surface (2.1) ,
it must be possible to rewrite equation (2.10) in the form

$$\frac{p}{\rho} = \pi G\rho a^2 c^3 A_3 [1 - \frac{\widetilde{\omega}^2}{a^2} - \frac{z^2}{c^2}] \ . \qquad (2.11)$$

Comparing the expressions (2.10) and (2.11), we conclude that

$$a^2 (A_1 - \frac{\Omega^2}{2\pi G\rho a^2 c}) = c^2 A_3 \ , \qquad (2.12)$$

i.e., by equations (2.9) ,

$$\Omega^2 = \frac{2\pi G\rho}{e^3}(1-e^2)^{\frac{1}{2}} [(3-2e^2) \sin^{-1} e - 3e(1-e^3)^{\frac{1}{2}}] \ . \qquad (2.12$$

III.   PERTURBATIONS ABOUT EQUILIBRIUM

Suppose that the equilibrium solution of Section II is infinitesimally

perturbed. The fluid velocity $\underset{\sim}{u}$ in the subsequent motion will satisfy the

incompressibility requirement

$$\text{div } \underset{\sim}{u} = 0 \quad , \tag{3.1}$$

and also the linearized Navier-Stokes equation

$$\frac{\partial \underset{\sim}{u}}{\partial t} + 2\underset{\sim}{\Omega} \times \underset{\sim}{u} = - \text{grad } \Pi + \nu \nabla^2 \underset{\sim}{u} \quad , \tag{3.2}$$

where ν denotes the kinematic viscosity, and

$$\Pi = \frac{p}{\rho} - V - \frac{1}{2} \Omega^2 \underset{\sim}{\omega}^2 \quad . \tag{3.3}$$

Thus $\underset{\sim}{u}$ must satisfy boundary conditions on the perturbed surface S of the

fluid. These require that the normal velocity of the fluid on S should equal

the normal velocity of the boundary S itself. They also require that the normal

components of the mechanical stress tensor should vanish on S . Introducing

the rate of strain tensor R_{ij}, defined by

$$R_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad , \tag{3.4}$$

the viscous stress tensor is $\nu \rho R_{ij}$. Since $\underset{\sim}{u}$ is a quantity of the first order

of smallness, the differences in the values of $R_{\zeta \theta}$ and $R_{\zeta \varphi}$ on S from their

value on S_e are of second order of smallness, and can therefore be neglected.

It follows that the demands that $\nu\rho R_{\zeta\theta}$ and $\nu\rho R_{\zeta\varphi}$ should vanish on S require

$$\text{either} \quad \nu = 0, \quad \text{or} \quad [R_{\zeta\theta}]_{S_e} = [R_{\zeta\varphi}]_{S_e} = 0 \ , \tag{3.5}$$

where we have used $[U]_V$ to denote the value of a quantity U on a surface V .

The remaining stress condition is

$$[p - 2\nu\rho R_{\zeta\zeta}]_S = 0 \tag{3.6}$$

and it is again convenient to extrapolate this condition back to the reference

surface S_e . For this purpose we write p and V in the forms

$$p = p_e + \delta p, \quad V = V_e + \delta V \ , \tag{3.7}$$

where p_e and V_e are the values of p and V in the equilibrium solution

(cf. eqs. [2.8] and [2.11]). We also introduce $\underset{\sim}{\xi}(\underset{\sim}{x}, t)$ the displacement of a

particle of fluid at time t from its equilibrium position $\underset{\sim}{x}$. Clearly we have

$$\underset{\sim}{u} = \partial\underset{\sim}{\xi}/\partial t \ , \tag{3.8}$$

to first order. Moreover, it is evident that

$$[p]_S \doteqdot [p_e]_S + [\delta p]_{S_e}$$

$$\doteqdot [p_e + \underset{\sim}{\xi} \cdot \text{grad } p_e]_{S_e} + [\delta p]_{S_e} \tag{3.9}$$

Thus condition (3.6) is equivalent to

$$[p_e + \underset{\sim}{\xi} \cdot grad\ p_e + \delta p - 2\nu \rho R_{\zeta \zeta}]_{S_e} = 0 \ , \qquad (3.10)$$

or, since equations (2.7) and (3.3) imply that

$$\Pi \doteq (\frac{p_e}{\rho} - V_e - \frac{1}{2}\Omega^2 \tilde{\omega}^2) + (\frac{\delta p}{\rho} - \delta V) = \frac{\delta p}{\rho} - \delta V + Const, \quad (3.11)$$

we have (absorbing the constant on the right-hand side of eq. [3.11])

$$[\Phi + \rho \Pi - 2\nu \rho R_{\zeta \zeta}]_{S_e} = 0 \ , \qquad (3.12)$$

where

$$\Phi = \rho \delta V + \underset{\sim}{\xi} \cdot grad\ p_e \qquad (3.13)$$

The significance of Φ may be seen by considering the changes δW in the gravitational energy W of the configuration

$$W = -\frac{1}{2} G\rho^2 \int_V \int_V \frac{d\underset{\sim}{x}\,d\underset{\sim}{x}'}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \ , \qquad (3.14)$$

consequent upon the deformation $\underset{\sim}{\xi}$. It is evident from equation (3.14) that

$$\delta V = -G\rho^2 \int_{V_e} \int_{V-V_e} \frac{d\underset{\sim}{x}\,d\underset{\sim}{x}'}{\underset{\sim}{x} - \underset{\sim}{x}'}$$

$$= -\rho \int_{V_e} \delta V(\underset{\sim}{x})\, d\underset{\sim}{x} \ . \qquad (3.15)$$

On the other hand, the work done against the pressure forces in the deformation ξ is, to first order,

$$-\int_{V_e} \xi \cdot \text{grad } p_e \, d\underset{\sim}{x} \quad . \tag{3.16}$$

Thus the sum of (3.15) and (3.16), namely $\int_{V_e} \Phi \, d\underset{\sim}{x}$, is the work required in order to subject the equilibrium configuration quasi-statically to the $\underset{\sim}{\xi}$ deformation. In the dynamical problem studied herein, it will equal the sum of two parts, namely, the decrease in kinetic energy and the energy degraded to heat by the action of viscosity.

IV.   GRAVITATIONAL EFFECTS

In calculating δV, the surface deformation may be divided into its harmonic components $P_n^m(\mu) \, e^{im\varphi}$, and each may be considered separately. We will write, therefore,

$$[\xi_\zeta]_{S_e} = \frac{1}{h_\zeta} K_{n,m} P_n^m(\mu) \, e^{im\varphi} \quad , \tag{4.1}$$

where

$$K_{n,m} = \frac{(2n+1)(n-m)!}{4\pi(n+m)!} \int_{-1}^{1} d\mu \int_{0}^{2\pi} d\varphi \, [L_\zeta \xi_\zeta]_{S_e} P_n^m(\mu) \, e^{-im\varphi} \tag{4.2}$$

The gravitational effect of this displacement may be likened to the addition of a surface mass distribution to S_e of density $\rho[\xi_\zeta]_{S_e}$. The gravitational

potential to which this gives rise must be a continuous solution of Laplace's equation, and must therefore be of the form

$$\delta V = \begin{cases} C_{n,m} \dfrac{Q_n^{\,m}(i\zeta)}{Q_n^{\,m}(iE)} \, P_n^{\,m}(\mu) \, e^{im\varphi} \, , & \zeta \geq E \, , \\[20pt] C_{n,m} \dfrac{P_n^{\,m}(i\zeta)}{P_n^{\,m}(iE)} \, P_n^{\,m}(\mu) \, e^{im\varphi} \, , & \zeta \leq E \, , \end{cases} \tag{4.3}$$

where $Q_n^{\,m}(i\zeta)$ denotes the associated Legendre function of the second kind of the orders n and m indicated, and of imaginary argument. For complex arguments, we adopt Hobson's definitions of $P_n^{\,m}(z)$ (see, for example, Hobson, 1931, § 54):

$$P_n^{\,m}(z) = (z^2 - 1)^{\frac{1}{2}m} \frac{d^m P_n(z)}{dz^m} \, , \qquad Q_n^{\,m}(z) = (z^2 - 1)^{\frac{1}{2}m} \frac{d^m Q_n(z)}{dz^m} \, .$$

The Wronskian relationship between these functions takes the form

$$P_n^{\,m}(z) \frac{d}{d_z} Q_n^{\,m}(z) - Q_n^{\,m}(z) \frac{d}{d_z} P_n^{\,m}(z) = \frac{(-)^m (n+m)!}{(n-m)!} \frac{1}{1-z^2} \tag{4.4}$$

(cf. Hobson, loc. cit., § 148).

In order to determine the constant $C_{n,m}$ appearing in the solution (4.3), we apply Gauss' theorem to an elementary volume containing the surface mass distribution:

$$\left[\frac{1}{h_\zeta}\frac{\partial}{\partial\zeta}\delta V\right]_{\zeta=E+0} - \left[\frac{1}{h_\zeta}\frac{\partial}{\partial\zeta}\delta V\right]_{\zeta=E-0} = -4\pi G\rho\left[\xi_\zeta\right]_{\zeta=E} \quad , \tag{4.5}$$

which, using equation (4.3), reduces to

$$C_{n,m}\left[\frac{1}{Q_n^{\,m}(i\zeta)}\frac{d}{d\zeta}Q_n^{\,m}(i\zeta) - \frac{1}{P_n^{\,m}(i\zeta)}\frac{d}{d\zeta}P_n^{\,m}(i\zeta)\right]_{\zeta=E} = -4\pi G\rho K_{n,m} \quad . \tag{4.6}$$

By using relation (4.4), we finally obtain

$$C_{n,m} = \frac{4\pi G\rho}{e^2}\frac{(-1)^m(n-m)!\,i}{(n+m)!}K_{n,m}\,P_n^{\,m}(iE)\,Q_n^{\,m}(iE) \quad . \tag{4.7}$$

It follows that

$$[\delta V]_{S_e} = \frac{4\pi G\rho^2}{e^2}\frac{i(-1)^m(n-m)!}{(n+m)!}K_{n,m}\,P_n^{\,m}(iE)\,Q_n^{\,m}(iE)\,P_n^{\,m}(\mu)\,e^{im\varphi} \quad . \tag{4.8}$$

In order to calculate $[\Phi]_{S_e}$, we note that, according to equation (2.11),

$$\mathrm{grad}\,p_e = -\frac{2}{e}G\rho^2\,a^3\,(1-e^2)A_3 h\,\underset{\sim}{1}_\zeta \quad , \tag{4.9}$$

where $\underset{\sim}{1}_\zeta$ is a unit vector in the direction of increasing ζ .

Thus, by equation (4.1) ,

$$[\underset{\sim}{\zeta}\cdot\mathrm{grad}\,p_e]_{S_e} = -\frac{2\pi}{e}G\rho^2\,a^3\,(1-e^2)A_3\,K_{n,m}\,P_n^{\,m}(\mu)\,e^{im\varphi} \tag{4.10}$$

Hence

$$[\Phi]_{S_e} = \frac{4\pi G\rho^2}{e^2}\left[\frac{i(-1)^m(n-m)!}{(n+m)!}P_n^{\,m}(iE)\,Q_n^{\,m}(iE) - \frac{1}{2}a^3\,e(1-e^2)A_3\right]K_{n,m}\,P_n^{\,m}(\mu)^{im\varphi}$$

$$\tag{4.11}$$

In the studies which follow, we will be particularly concerned with the case $n = m = 2$. For this harmonic, we have

$$P_2^{\ 2}(iE) = -3(1+E^2) \quad , \quad Q_2^{\ 2}(iE) = i[3(1+E^2)\cot^{-1}E - \frac{(5E+3E^3)}{(1+E^2)}] \qquad (4.12)$$

and Φ takes the form

$$[\Phi]_{S_e} = -\frac{\pi G\rho^2}{2e^3}[e(1-e^2)^{\frac{1}{2}}(3+10e^2)-(3+8e^2-8e^4)\sin^{-1}e]K_{2,2}P_2^2(\mu)e^{2i\varphi} \quad .$$

$$(4.13)$$

V.   THE INVISCID SOLUTIONS

Cartan (1922) and others (see, for example, Bryan, 1888) have examined the case $\nu = 0$ in great detail, and have shown that all the normal modes of oscillation can be generated from polynomial solutions of Poincare's differential equation[*]:

$$[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (1 + \frac{4\Omega^2}{s^2})\frac{\partial^2}{\partial z^2}] = 0 \quad , \qquad (5.1)$$

where $s = \partial/\partial t$. Since boundary conditions (3.5) are automatically satisfied when $\nu = 0$, there is only one condition (viz. eq. [3.12]) to be applied on S_e, and this determines s^2 for the normal mode considered. Although

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[*] That Poincare's equation arises may be seen simply by solving eq. [3.2] for $\underset{\sim}{u}$ in terms of grad Π and substituting the result into eq. [3.1].

equation (5.1) is hyperbolic when $-4\Omega^2 < s^2 < 0$ and the boundary condition (3.12) is the Dirichlet type, Cartan was able to establish that, even in this case, the polynomial solutions are unique.

We can illustrate the solution simply for the important case

$$\Pi = h(x+iy)^2 , \quad (h = \text{constant}) , \tag{5.2}$$

which evidently satisfies equation (5.1). From equation (3.2) we find

$$ux = -\frac{2h(x+iy)}{(s-2i\Omega)} , \quad uy = -\frac{2h(x+iy)}{(s-2i\Omega)} , \quad u_z = 0 . \tag{5.3}$$

It should be noticed in passing that this solution satisfies

$$\nabla^2 \underset{\sim}{u} = 0 . \tag{5.4}$$

According to equation (5.3), u_ζ is given by

$$h_\zeta u_\zeta \equiv a^2 e(1-e^2)^{\frac{1}{2}} (\frac{x}{a^2} u_x + \frac{y}{a^2} u_y + \frac{z}{c^2} u_z)$$

$$\tag{5.5}$$

$$= -\frac{2e^3(1-e^2)^{\frac{1}{2}}}{(s-2i\Omega)} h(x+iy)^2 .$$

Thus, on S_e we have

$$[\Pi]_{S_e} = \frac{1}{3} a^2 h P_2^2 (\mu) e^{2i\varphi} \tag{5.6}$$

$$[h_\zeta u_\zeta]_{S_e} = -\frac{2a^2 he(1-e^2)^{\frac{1}{2}}}{3(s-2i\Omega)} P_2^2 (\mu) e^{2i\varphi} . \tag{5.7}$$

From equation (5.7), in the notation of equation (4.1), we find

$$K_{2,2} = - \frac{2a^2 he(1-e^2)^{\frac{1}{2}}}{3s(s-2i\Omega)} .$$ (5.8)

Thus drawing together equations (5.6), (5.8), (4.13) and (3.12), we find

$$\frac{1}{3}\rho a^2 h + \frac{2a^2 e(1-e^2)^{\frac{1}{2}}}{3s(s-2i\Omega)} \frac{h\pi G\rho^2}{2e^2} [e(1-e^2)^{\frac{1}{2}}(3+10e^2)-(3+8e^2-8e^4)\sin^{-1}e] = 0 ,$$

i.e.

$$s(s-2i\Omega) = - \frac{\pi G\rho}{e^5}(1-e^2)^{\frac{1}{2}} [e(1-e^2)^{\frac{1}{2}}(3+10e^2)-(3+8e^2-8e^4)\sin^{-1}e] .$$ (5.9)

This is Cartan's result (loc. cit. p. 338). Writing it in the form

$$s(s-2i\Omega) + Q = 0$$ (5.10)

we find that either $s = s_a$ or $s = s_b$ where

$$s_a = i[\Omega - (Q + \Omega^2)^{\frac{1}{2}}] = in_a(\text{say}) ,$$ (5.11a)

$$s_b = i[\Omega + (Q + \Omega^2)^{\frac{1}{2}}] = in_b(\text{say}) .$$ (5.11b)

At the point of bifurcation of the Maclaurin sequence, Q is zero, the root (5.11a) is therefore zero, and the spheroid is neutrally stable. Beyond bifurcation, s_a is again non-zero, and the spheroid is again stable. However, if the argument of Thomson and Tait is correct (see Section I), viscosity should introduce into root (5.11a) a real part which changes sign from negative to positive on passing through the point of bifurcation. This possibility is studied in the next section.

Before leaving this solution it is useful to compute the rate of strain

tensor. According to equations (3.4) and (5.3) we find that, in the cartesian

coordinate system (x, y, z)

$$R_{ij} = -\frac{2h}{(s-2i\Omega)} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (5.12)$$

The rate of strain is thus constant throughout the spheroid. It may be written

simply in dyadic form

$$R_{ij} = -\frac{2h}{(s-2i)} q_i q_j \qquad (5.13)$$

where q is a vector whose (x, y, z) components are

$$q = (1, i, 0) , \qquad (5.14)$$

and whose (ζ, θ, φ) components are therefore

$$q = (\sin \psi, \cos \psi, i) e^{i\varphi} , \qquad (5.15)$$

where

$$\cos \psi = \frac{\Omega_\zeta}{\Omega} = \mu \left(\frac{1+\zeta^2}{\zeta^2+\mu^2}\right)^{\frac{1}{2}} , \quad \sin \psi = \frac{\Omega\widetilde{\omega}}{\Omega} = \left(\frac{1-\mu^2}{\zeta^2+\mu^2}\right)^{\frac{1}{2}} \qquad (5.16)$$

By equations (5.13) and (5.15), we find immediately that

$$R_{\zeta\zeta} = -\frac{2h\sin^2\psi\,e^{2i\varphi}}{(s-2i\Omega)}, \quad R_{\zeta\theta} = \frac{2h\sin\psi\cos\psi\,e^{2i\varphi}}{(s-2i\Omega)},$$

$$\tag{5.17}$$

$$R_{\zeta\varphi} = -\frac{2hi\sin\psi\,e^{2i\varphi}}{(s-2i\Omega)}.$$

VI.   THE EFFECT OF A SMALL VISCOSITY

The solution obtained in Section V for an inviscid fluid can, in principle, be modified in two ways when the fluid has a small kinematic viscosity ν . First, the equations of motion with $\nu = 0$ may not be satisfied by the inviscid solution so that correction terms of order ν are required. In the present problem, however, in which we are concerned with the solution (5.3), the viscous terms of equation (3.2) vanish identically in virtue of equation (5.4), and no such modifications of the inviscid solution are required. Second, the boundary conditions which must be satisfied by the solution for a viscous fluid include the requirements (cf. eq. [3.5]) that the tangential components of R_{ij} must vanish at the free surface S . If, as in the present problem, these conditions are not satisfied by the inviscid solution, then a boundary layer in which viscous effects are significant must be introduced in order to make the necessary adjustments to the tangential stresses. We suppose that the leading terms in the boundary layer vanish at its inner edge so that the inviscid solution is not modified to the order of these terms. However, in satisfying the conditions on the tangential stress components, the condition on the normal component is

invalidated, albeit to a higher order in ν . In order to correct this, all the

solutions of the inviscid problem involving the surface harmonics for which

$m = 2$ must be excited, including the one with $n = 2$. Equation (5.9) is

consequently modified.

For small values of the viscosity $(\nu << a^2 \Omega)$ it is accordingly convenient

to proceed as follows. Write

$$\underset{\sim}{u} = \underset{\sim}{u}_0 + \underset{\sim}{u}_1 \ , \qquad \Pi = \Pi_0 + \Pi_1 \ , \qquad \Phi = \Phi_0 + \Phi_1 \ , \qquad (6.1)$$

where $\underset{\sim}{u}_0$ and Π_0 are solutions of the inviscid equations with $m = n = 2$

but which do not obey equation (5.9), and $\underset{\sim}{u}_1$ and Π_1 , are the leading terms

of the modification due to viscosity which vanish in the interior of the fluid.

In addition Φ_0 is the value of Φ arising from the perturbation of the free

surface due to $\underset{\sim}{u}_0$, with a corresponding definition for Φ_1 . Then, from

equation (3.12), we have

$$[\Phi + \rho\Pi - 2\nu\rho R_{\zeta\zeta}]_{S_e} = 0 \ , \qquad (6.2)$$

i.e.

$$[\Phi_0 + \rho\Pi_0]_{S_e} + [\Phi_1 + \rho\Pi_1 - 2\nu\rho R_{0\zeta\zeta}]_{S_e} = 0 \ , \qquad (6.3)$$

as the condition on the normal stress to be satisfied by the solution. We are

particularly interested in the harmonic $n = m = 2$, and so we multiply equation

(6.3) by $P_2^2(\mu) e^{-2i\varphi}$ and integrate over the surface of the spheroid, obtaining

$$\frac{1}{3}\rho a^2 h[1 + \frac{Q}{s(s-2i\Omega)}] + \gamma = 0 \qquad (6.4)$$

where we have used the theory of Section V. Here γ is the contribution from the second term of equation (6.3):

$$\gamma = \frac{5}{32\pi} \int_{-1}^{1} du \int_{0}^{2\pi} d\varphi [\Phi_1 + \rho\Pi_1 - 2\nu\rho R_{0\zeta\zeta}]_{S_e} (1-\mu^2) e^{-2i\varphi} . \qquad (6.5)$$

In the normal mode theory, it is assumed that time enters the dependent variables through the factor e^{st} , so that

$$[h_\zeta \xi_\zeta]_{S_e} = \frac{1}{s}[h_\zeta u_\zeta]_{S_e} . \qquad (6.6)$$

Using equations (4.1) and 4.13) and the definition of Q (cf. eqs. [5.9] and [5.10]), it follows that

$$\gamma = \frac{5\rho}{32\pi} \int_{-1}^{1} d\mu \int_{0}^{2\pi} d\varphi [\Pi_1 - \frac{(1+E^2)Q}{2sE}(h_\zeta u_{1\zeta}) - 2\nu R_{0\zeta\zeta}]_{S_e} (1-\mu^2) e^{-2i\varphi} . \qquad (6.7)$$

Our main purpose here is to evaluate γ .

An examination of equations (3.1), (3.2) and (3.5) suggests that we **assume** that the boundary layer is of the conventional type in which

$$u_{1\theta} = O(\nu^{\frac{1}{2}}), \qquad u_{1\varphi} = O(\nu^{\frac{1}{2}}), \qquad \partial/\partial\zeta = O(\nu^{-\frac{1}{2}}) ,$$

$$\partial/\partial\theta = O(1), \qquad \partial/\partial\varphi = O(1) , \qquad (6.8)$$

so that

$$u_{1\zeta} = O(\nu), \qquad \Pi_1 = O(\nu) \qquad (6.9)$$

all orders of magnitude referring to the dependence upon ν .

The leading terms of the θ, φ and ζ components of the equations of momentum (3.2) and the complete equation of continuity (3.1) may then be written

$$(\frac{\nu}{h_\zeta^2}\frac{\partial^2}{\partial\zeta^2} - s)u_{1\theta} + 2\Omega\cos\psi\, u_{1\varphi} = 0 \ , \tag{6.10}$$

$$(\frac{\nu}{h_\zeta^2}\frac{\partial^2}{\partial\zeta^2} - s)u_{1\varphi} - 2\Omega\cos\psi\, u_{1\theta} = 0 \ , \tag{6.11}$$

$$\frac{1}{h_\zeta}\frac{\partial\Pi_1}{\partial\zeta} = 2\Omega\sin\psi\, u_{1\varphi} \ , \tag{6.12}$$

$$(1+E^2)h_\zeta\frac{\partial u_{1\zeta}}{\partial\zeta} - \frac{\partial}{\partial\mu}[(1-\mu^2)^{\frac{1}{2}}h_\theta\, u_{1\theta}] + \frac{2ik(E^2+\mu^2)u_{1\varphi}}{(1+E^2)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}} = 0 \ . \tag{6.13}$$

In all of these equations h_ζ and h_θ are to be evaluated at $\zeta = E$, and are therefore functions of μ alone. Further, in equation (6.13), we have made use of the property that all dependent variables depend on φ, only though the factor $e^{2i\varphi}$. Combining equations (6.12) and (6.13) we obtain

$$\frac{\partial}{\partial\zeta}\{(1-\mu^2)[\Pi_1 - \frac{(1+E^2)Q}{2sE}h_\zeta\, u_{1\zeta}]\}$$

$$= 2\, h_\zeta\sin\psi(1-\mu^2)u_{1\varphi} + \frac{Qh_\zeta}{sE^2}(E^2+\mu^2)\sin\psi\,(iu_{1\varphi} - \cos\psi\, u_{1\theta})$$

$$- \frac{kQ}{2sE}\frac{\partial}{\partial\mu}[(1-\mu^2)^{3/2}(E^2+\mu^2)^{\frac{1}{2}}u_{1\theta}] \ . \tag{6.14}$$

Anticipating that γ is of order ν, we may assume for the purposes of

simplifying equation (6.14) that $Q = -s(s - 2i\Omega)$ (cf. eq. 5.10), whence

$$\frac{\partial}{\partial\zeta}\{(1-\mu^2)[\Pi_1 - \frac{(1+E^2)Q}{2sE} h_\zeta u_{1\zeta}]\}$$

$$= \frac{(E^2+\mu^2)h_\zeta \sin\psi}{2E^2} [(u_{1\theta}-iu_{1\phi})(1+\cos\psi)(s-2i\Omega)$$

$$-(u_{1\theta}+iu_{1\phi})(1-\cos\psi)(s+2i\Omega\cos\psi)]$$

$$+ \frac{k(s-2i\Omega)}{2E} \frac{\partial}{\partial\mu} [(1-\mu^2)^{3/2}(E^2+\mu^2)^{\frac{1}{2}} u_{1\theta}] \quad . \tag{6.15}$$

The values of $u_{1\theta}$ and $u_{1\varphi}$ required for the evaluation of the right-hand

side of equation (6.15) follow from equations (6.10) and (6.11). Write

$$\frac{\nu\alpha^2}{h_\zeta^2} = s+2i\Omega\cos\psi \quad , \qquad \frac{\nu\beta^2}{h_\zeta^2} = s-2i\Omega\cos\psi \quad , \tag{6.16}$$

where the real parts of α and β are both positive. The general solution of

equations (6.10) and (6.11) which vanishes when $\nu^{-\frac{1}{2}}(E-\zeta)$ is large and positive

is then

$$u_{1\theta} = \frac{1}{2} Ae^{\alpha(\zeta-E)} + \frac{1}{2} Be^{\beta(\zeta-E)} \quad ,$$

$$u_{1\varphi} = \frac{1}{2} iAe^{\alpha(\zeta-E)} + \frac{1}{2} iBe^{\beta(\zeta-E)} \quad . \tag{6.17}$$

The values of the coefficient $A(\mu, \varphi)$ and $B(\mu, \varphi)$ are found from the condition that the tangential components of stress vanish at $\zeta = E$. Using equations (5.17), we have

$$\frac{1}{4h_\zeta}(\alpha A + \beta B) = \frac{2h \sin\psi \cos\psi}{s - 2i\Omega} e^{2i\varphi} , \tag{6.18}$$

$$\frac{1}{4h_\zeta}(\alpha A - \beta B) = -\frac{2ih \sin\psi}{s - 2i\Omega} e^{2i\varphi} ,$$

and consequently

$$A = \frac{4khE}{\alpha(1 + E^2)^{\frac{1}{2}}(s - 2i\Omega)} (\cos\psi - 1)(1 - \mu^2)^{\frac{1}{2}} e^{2i\varphi}$$

$$\tag{6.19}$$

$$B = \frac{4khE}{\beta(1 + E^2)^{\frac{1}{2}}(s - 2i\Omega)} (\cos\psi + 1)(1 - \mu^2)^{\frac{1}{2}} e^{2i\varphi}$$

It is noted that the a priori assumptions (6.8) about the boundary layer have led to a reasonable solution which may easily be shown to be consistent provided neither α nor β vanishes, i.e. if we write

$$s = 2i\Omega \cos\psi_0 , \qquad (0 < \psi_0 < \Pi)^* \tag{6.20}$$

provided we exclude the neighborhood of the singularities $\psi = \psi_0$ and $\psi = \Pi - \psi_0$

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* ψ_0 is real for the range $0.4285 < e < 0.9529 \ldots$ of eccentricity. Since this range contains the point of bifurcation, the singularities defined by equations (6.20) are germane to our discussion.

These zones require special treatment which we shall postpone until the next

section. We merely observe here that their contribution to γ is of order $\nu^{\frac{6}{5}}$.

Substituting the results (6.17) and (6.19) into the first term appearing on

the right-hand side of equation (6.15), integrating with respect to ζ from

$-\infty$ to 0, and noting that Π, and $u_{1\zeta}$ vanish in the interior of the fluid, we

find that

$$(1-\mu^2)[\Pi_1 - \frac{(1+E^2)Q}{2sE} h_\zeta u_{1\zeta}]_{S_e}$$

$$= \frac{2h_\zeta h\nu(1+E^2)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}\sin\psi\, e^{2i\varphi}}{kE(s-2i\Omega)} [(1+\cos\psi)^2 + (1-\cos\psi)^2]$$

$$+ \int_{-\infty}^{0} \frac{h(s-2i\Omega)}{2E} \frac{\partial}{\partial\mu} [(1-\mu^2)^{\frac{1}{2}}(E^2+\mu^2)^{\frac{1}{2}} u_{1\theta}]d\zeta \quad . \tag{6.21}$$

The determination of γ is completed by using the form of $R_{0\zeta\zeta}$ given by

equation (5.17)

$$[2 R_{0\zeta\zeta}]_{S_e} = - \frac{4h\nu E^2 (1-\mu^2) e^{2i\varphi}}{(E^2+\mu^2)(s-2i\Omega)} \quad , \tag{6.22}$$

and adding it to equation (6.21) giving

$$\rho(1-\mu^2) [\Pi_1 - \frac{(1+E^2)Q}{2sE} h_\zeta u_\zeta - 2\nu R_{0\zeta\zeta}]_{S_e}$$

$$\tag{6.23}$$

$$= \frac{8h\nu\rho}{(s-2i\Omega)} (1-\mu^2) e^{2i\varphi} + \int_{-\infty}^{0} \frac{\rho k(s-2i\Omega)}{2E} \frac{\partial}{\partial\mu} [(1-\mu^2)^{\frac{1}{2}}(E^2+\mu^2)^{\frac{1}{2}} u_{1\theta}] d\zeta \quad .$$

Carrying out the integration (6.5), assuming that the orders of integration with respect to ζ and μ can be inverted, we obtain

$$\gamma = \frac{10h\nu\rho}{3(s-2i\Omega)} \quad , \tag{6.24}$$

and hence, from equation (6.4),

$$1 + \frac{Q}{s(s-2i\Omega)} + \frac{10}{a^2(s-2i\Omega)} = 0 \tag{6.25}$$

is the modified dispersion relationship for the harmonic $n = m = 2$, neglecting powers of ν above the first. Let one of its roots be

$$s = s_0 + s_1 \tag{6.26}$$

where s_0 is one of the roots (5.11) and s_1 is of order ν . On substituting into equation (6.25) and neglecting all powers of s_1 above the first, we find that

$$s_1 = -\frac{5\nu s_0}{a^2(s_0-2i\Omega)} = \frac{5\nu s_0^2}{a^2 Q} \quad . \tag{6.27}$$

For small values of Ω, equation (5.11) implies that $Q \doteq -s_0^2$. Thus, according to equation (6.27) $s_1 \doteq -5\nu/a^2$, in agreement with the result of Lamb (1881) for the oscillations of a slightly viscous sphere (see also Chandrasekhar, 1959). Since s_0^2 is always negative up to the point of ordinary instability (e $\doteq$ 0.9529), the stability of the slightly viscous spheroid depends on the sign of Q . Now Q is positive between $e = 0$ (the

sphere) and $e \doteq 0.8127$ (where the Maclaurin sequence bifurcates) and is

negative between this value of e and $e \doteq 0.9529$. It follows that, beyond

bifurcation, the value of s_1 is real and positive so that oscillations of the

Maclaurin spheroid in which the harmonic $n = m = 2$ are excited increase

exponentially in amplitude with time in agreement with the conclusion based on

the energy argument of Section I. Presumably once this instability sets in,

the configuration moves towards the lower energy Jacobi state with the same

angular momentum.

VII.   THE SINGULAR CIRCLES

The theory of the boundary layer discussed in the previous section failed

in the neighborhood of the circles of latitude $\psi = \psi_0$, and $\psi = \Pi - \psi_0$, where

ψ_0 is defined by equation (6.20). One still expects the boundary layer to be

vanishingly thin and the velocities in it to be vanishingly small as $v \to 0$, but

the hypothesis (6.8) on which the theory is based needs modification. Let us

concentrate attention on the zone R_+ in the neighborhood of $\psi = \psi_0$ and

suppose that, instead of assumption (6.8),

$$\partial/\partial\zeta = O(v^{-p}), \qquad \partial/\partial\mu = O(v^{-q}) , \qquad (7.1)$$

where $1/2 > p > q > 0$; thus we expect the boundary layer to be thicker in R_+

and rapid changes to occur in the μ-direction but that the most rapid changes

still occur in the ζ-direction. It will be shown that $p = 2/5$, $q = 1/5$ leads

to a consistent picture.

Further we shall assume that

$$u_{1\theta} = O(\nu^r), \qquad u_{1\varphi} = O(\nu^r) , \qquad (7.2)$$

where $r > 0$. The equation of continuity (6.13) may now be written as

$$\frac{1}{h_\zeta} \frac{\partial u_{1\zeta}}{\partial \zeta} - \frac{(1-\mu^2)^{\frac{1}{2}}}{h_\theta} \frac{\partial u_{1\theta}}{\partial \mu} = 0 \qquad (7.3)$$

where h_ζ and h_θ are now constants, so that $u_{1\zeta} = O(\nu^{r+p-q})$. The equation of momentum in the θ-direction reduces to

$$\frac{\nu}{h_\zeta^2} \frac{\partial^2 u_{1\theta}}{\partial \zeta^2} = su_{1\theta} - 2\Omega \cos\psi\, u_{1\varphi} - \frac{(1-\mu^2)^{\frac{1}{2}}}{h_\theta} \frac{\partial \Pi_1}{\partial \mu} ; \qquad (7.4)$$

notice that in R_+, $su_{1\theta} \doteq 2\Omega\cos\psi\, u_{1\varphi}$ by equations (6.16) and (6.17). It follows from equation (7.4) that $\Pi_1 = O(\nu^{1+r-2p+q})$. The equation of momentum in the φ-direction reduces to

$$\frac{\nu}{h_\zeta^2} \frac{\partial^2 u_{1\varphi}}{\partial \zeta^2} = su_{1\varphi} + 2\Omega(\cos\psi\, u_{1\theta} + \sin\psi\, u_{1\zeta}) , \qquad (7.5)$$

the pressure terms being of smaller order than the other terms of equation (7.5) since $\partial\Pi_1/\partial\varphi = 2i\Pi_1$. The equation of momentum in the ζ-direction is

$$\frac{1}{h_\zeta} \frac{\partial \Pi_1}{\partial \zeta} = 2\Omega u_{1\varphi} \sin\psi , \qquad (7.6)$$

the viscous terms being negligible because $\partial\Pi_1/\partial\zeta \gg \partial\Pi_1/\partial\mu$ in virtue of

assumption (7.1), and $u_{1\zeta} \ll u_{1\phi}$ in virtue of assumption (7.2) and equation

(7.3). It is to be particularly noted that this equation is identical with

equation (6.12), and that both the equations used to derive the basic

result (6.14) hold also in R_+ . Consequently, our aim here is to find the

contribution of R_+ to equation (6.14). If we now eliminate Π_1 and $u_{1\zeta}$ from

equations (7.3) to (7.6), and write

$$Z_+ = u_{1\theta} - iu_{1\varphi} \qquad (7.7)$$

we find that

$$\frac{\nu}{h_\zeta^2} \frac{\partial^3 Z_+}{\partial\zeta^3} = (s - 2i\Omega\cos\psi) \frac{\partial Z_+}{\partial\zeta} - 2i\Omega\sin\psi \frac{h_\zeta}{h_\theta}(1-\mu^2)^{\frac{1}{2}} \frac{\partial Z_+}{\partial\mu} . \qquad (7.8)$$

Similarly, if

$$Z_- = u_{1\theta} + iu_{1\varphi} \qquad (7.9)$$

$$\frac{\nu}{h_\zeta^2} \frac{\partial^3 Z_-}{\partial\zeta^3} = (s + 2i\Omega\cos\psi) \frac{\partial Z_-}{\partial\zeta} + 2i\Omega\sin\psi \frac{h_\zeta}{h_\theta}(1-\mu^2)^{\frac{1}{2}} \frac{\partial Z_-}{\partial\mu} . \qquad (7.10)$$

Thus, in equation (7.8), the last term on the right-hand side may be neglected

except in R_+, and, in equation (7.9), the last term may be neglected except

in R_-, the neighborhood of $\psi = \pi - \psi_0$. Concentrating attention on R_+ ,

let us write $\psi = \psi_0 + \Psi$ where Ψ is small. Then equation (7.8) simplifies

to

$$\frac{\nu}{h_\zeta{}^2} \frac{\partial^3 Z_+}{\partial \zeta^3} = 2i\Omega \sin\psi_0 \ \Psi \frac{\partial Z_+}{\partial \zeta} - \frac{2i\Omega}{kE} h_\zeta \sin\psi_0 \frac{\partial Z_+}{\partial \Psi} \ . \qquad (7.10)$$

From a comparison of the orders of magnitude of the three terms composing equation (7.10), and by using the assumed properties (7.1), we have

$$a = \frac{2}{5} \ , \qquad b = \frac{1}{5} \ . \qquad\qquad (7.11)$$

Outside R_+

$$Z_+ = Be^{\beta(\zeta - E)} \ , \qquad\qquad (7.12)$$

by equations (6.17). Consequently on the edge of R_+ where $|\psi - \psi_0|$ is small but $\nu^{-1/5} \Psi$ is large,

$$Z_+ \doteq \frac{B_1 \nu^{\frac{1}{2}}}{|\Psi|^{\frac{1}{2}}} \ \exp\left[\{2i\Omega \ h_\zeta{}^2 \ \sin\psi_0 \ \frac{\Psi}{\nu}\}^{\frac{1}{2}} (\zeta - E)\right] , \qquad (7.13)$$

where B_1 is a factor of order unity. Comparing the orders of magnitude, it follows that

$$Z_+ = O(\nu^{2/5}) \qquad\qquad (7.14)$$

in R_+ . Moreover $\partial Z_+/\partial \zeta = O(1)$ in R_+, as it must be since it is required to play a part in the cancellation of the tangential stresses. Although it has not been explicitly evaluated, the determination of Z_+ in R_+ seems to be a properly posed problem since, for not only do we have the behaviors (7.13) to determine the relevant solution of equation (7.10) at the edges of R_+, but we

also have $Z_+ \to 0$ as $\zeta \to -\infty$ and, finally, $\partial Z_+ / \partial_\zeta$ = known stress on $\zeta = E$.

Referring back to Section VI, the presence of the singular lines led to two apparent difficulties peculiar to the boundary layers studied here. First, it was not clear that the orders of integration and differentiation can be inverted as was done in evaluating the integral (6.5) from the result (6.23). Now that the behavior of Z_+ in R_+ and, by implication, of Z_- in R_- is seen to be satisfactory, we can justify the inversion by dividing up the interval $-1 \leq \mu \leq 1$ into three regions by separating out R_- and R_+ . In none of these three regions does the inversion present any difficulty. Second, a question arose concerning the contribution from R_+ and R_- to γ via the first term on the right-hand side of equation (6.15). The contribution from R_+ is easily seen to be of the same order in ν as

$$\iiint d\varphi \, d\zeta \, d\Psi \, Z_+ (s - 2i\Omega \cos \psi)$$

i.e. as

$$\iiint d\varphi \, d\zeta \, d\Psi \, Z_+ \, \Psi \tag{7.15}$$

and, using the results (7.11) and (7.14), this integral is of order $\nu^{6/5}$. Thus we see that, although the velocities in the neighborhood of R_+ are larger by a factor of $\nu^{-1/10}$ than those elsewhere in the boundary layer, their contribution to γ is smaller by a factor of $\nu^{1/5}$. A similar remark applies to R_- .

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